PHOTON GREEN FUNCTIONS IN CURVED SPACETIME

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Quantization of electrodynamics in curved space-time in the Lorenz gauge and with arbitrary gauge parameter makes it necessary to study Green functions of non-minimal operators with variable coefficients. Starting from the integral representation of photon Green functions, we link them to the evaluation of integrals involving Γ-functions. Eventually, the full asymptotic expansion of the Feynman photon Green function at small values of the world function, as well as its explicit dependence on the gauge parameter, are obtained without adding by hand a mass term to the Faddeev–Popov Lagrangian. Coincidence limits of second covariant derivatives of the associated Hadamard function are also evaluated, as a first step towards the energy-momentum tensor in the non-minimal case.

1 Introduction

Recent investigations of the force acting on a rigid Casimir apparatus in a gravitational field [1][2] have provided another relevant example of how Green function methods [3] can prove useful in modern gravitational physics. This has motivated our interest in a thorough analysis of the photon Green functions in curved space-time, starting with the case of curved manifolds without boundary. Our results are a concise formula for the local asymptotics of the Feynman photon Green function in the case of a non-minimal operator acting on the electromagnetic potential, and an expression for second covariant derivatives of the associated Hadamard function in the coincidence limit. The latter can be used, in turn, to study the regularized energy-momentum tensor. All of our analysis is performed by using ζ-function regularizaton without adding by hand a mass term to the BRST-invariant action, which is indeed desirable from the point of view of general principles in quantum field theory: a mass term added by hand spoils gauge invariance of the Maxwell part of the action, nor does it respect BRST invariance of the full action. Moreover, the addition by hand of a term $\frac{m^2}{2}A_\mu A^\mu$ leads to a bad ultraviolet behaviour of the photon propagator in momentum space. This is taken care of by introducing an auxiliary vector field, but then unitarity of the quantum theory is lost. [4]

2 From the heat kernel to the Green function

We consider quantum Maxwell theory in curved space-time via path integrals, hence adding gauge-fixing term and ghost-fields (here χ, ψ) contribution. The full action reads therefore as (hereafter $g = -\det g_{\mu\nu}(x)$)

$$S = \int d^4 x \sqrt{g} \left[-\frac{1}{4} g^{\mu\rho} g^{\nu\beta} F_{\mu\nu} F_{\rho\beta} - \frac{(\nabla^\mu A_\mu)^2}{2\alpha} - \frac{\chi}{\sqrt{\alpha}} \Box \psi \right]. \tag{1}$$

This leads to the gauge-field operator

$$P^{\mu\nu}(\alpha) = -g^{\mu\nu} \Box + R^{\mu\nu} + \left(1 - \frac{1}{\alpha}\right) \nabla^\mu \nabla^\nu, \tag{2}$$

with associated 'heat kernel' obeying the first-order equation

$$\mathrm{i} \frac{\partial}{\partial \tau} K^{(\alpha)}_{\mu\nu'}(\tau) = P_\mu{}^\lambda(\alpha) K^{(\alpha)}_{\lambda\nu'}(\tau), \tag{3}$$

with initial condition $K^{(\alpha)}_{\mu\nu'}(\tau = 0) = g_{\mu\nu}(x)\delta(x, x')$. Thanks to the work of Endo, [5] the heat kernel for arbitrary values of α can be obtained from the heat kernel at $\alpha = 1$ through the formula

$$K^{(\alpha)}_{\mu\nu'}(\tau) = K^{(1)}_{\mu\nu'}(\tau) + \mathrm{i} \int_\tau^{\tau/\alpha} dy \nabla_\mu \nabla^\lambda K^{(1)}_{\lambda\nu'}(y). \tag{4}$$

The Feynman photon Green function is eventually obtained from the definition

$$g^{\frac{1}{4}} G^{(\alpha)}_{\mu\nu'} g'^{\frac{1}{4}} \equiv \lim_{s \to 0} \frac{\mu_A^{2s} \mathrm{i}^{s+1}}{\Gamma(s+1)} \int_0^\infty d\tau \, \tau^s K^{(\alpha)}_{\mu\nu'}(\tau), \tag{5}$$

where the limit as $s \to 0$ should be taken *at the end of all calculations*, and the regularization involving τ^s is necessary since we do not rotate the integration contour nor do we add by hand the $\frac{m^2}{2} A_\mu A^\mu$ term to the action as we said before. By virtue of Eqs. (4) and (5), we can evaluate the local asymptotics of the Feynman Green function by exploiting the Fock–Schwinger–DeWitt local asymptotics for the heat kernel $K^{(1)}_{\mu\nu'}(\tau)$, i.e.

$$K^{(1)}_{\mu\nu'}(\tau) \sim \frac{\mathrm{i}}{16\pi^2} g^{\frac{1}{4}} \sqrt{\triangle} g'^{\frac{1}{4}} \mathrm{e}^{\frac{i\sigma}{2\tau}} \sum_{n=0}^\infty (\mathrm{i}\tau)^{n-2} b_{n\,\mu\nu'}, \tag{6}$$

where $\sigma = \sigma(x, x')$, called the world function, is half the square of the geodesic distance between x and x', the bi-scalar $\sqrt{\triangle}(x, x')$ is defined by the equation

$$\sqrt{g} \; \triangle \; \sqrt{g'} = \det \sigma_{;\mu\nu'}, \tag{7}$$

and the coefficient bi-vectors $b_{n\,\mu\nu'}$ are evaluated, in principle, from the recursion formula

$$\sigma^{;\lambda} b_{n\,\mu\nu';\lambda} + n b_{n\,\mu\nu'} = \frac{1}{\sqrt{\triangle}} \left(\sqrt{\triangle} b_{n-1,\mu\nu'} \right)_{;\lambda}{}^\lambda - R_\mu{}^\lambda \, b_{n-1,\lambda\nu'}, \tag{8}$$

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obtained substituting the asymptotic expansion (6) in Eq. (3). On using Eqs. (4)–(6) and the formulae (suitable restrictions on the real parts of the parameters are taken to hold, to ensure existence of our integrals [6])

$$\int_0^\infty y^{-\beta} e^{iy} dy = i\Gamma(1-\beta)e^{-i\frac{\pi}{2}\beta}, \tag{9}$$

$$\int_0^\infty x^{\beta-1}\Gamma(\nu, cx)dx = \frac{\Gamma(\beta+\nu)}{\beta c^\beta}, \tag{10}$$

we eventually find, in the limit as $\sigma(x, x') \to 0$, the local asymptotics of the Feynman Green function in the form

$$G^{(\alpha)}_{\mu\nu'} \sim \frac{i}{16\pi^2} \lim_{s\to 0} \frac{\mu_A^{2s}}{\Gamma(s+1)} \mathcal{G}^{(\alpha)}_{\mu\nu'}(s), \tag{11}$$

where, after having defined

$$U_{n\;\mu}{}^\lambda(s;\alpha) \equiv \frac{2}{\sigma(x,x')}\delta_\mu{}^\lambda + \frac{(\alpha^{s+1}-1)}{(s+n)(s+1)}\nabla_\mu\nabla^\lambda, \tag{12}$$

$$B_{n\;\lambda\nu'}(s) \equiv b_{n\;\lambda\nu'}\sqrt{\triangle}(x,x')(\sigma(x,x')/2)^{s+n}, \tag{13}$$

we write

$$\mathcal{G}^{(\alpha)}_{\mu\nu'}(s) \equiv \sum_{n=0}^\infty \Gamma(1-s-n)U_{n\;\mu}{}^\lambda(s;\alpha)B_{n\;\lambda\nu'}(s). \tag{14}$$

3 Singularities in the local asymptotics

Since $b_{0\;\mu\nu'} = g_{\mu\nu'}$, i.e. the parallel displacement matrix, we recover immediately, from Eqs. (11)–(14), the term

$$\frac{i}{8\pi^2}\frac{\sqrt{\triangle}(x,x')}{\sigma(x,x')+i\varepsilon}g_{\mu\nu'}$$

in the local asymptotics of $G^{(\alpha)}_{\mu\nu'}$. Moreover, on setting $\alpha = 1$ for simplicity, the infinite sum (14) can be evaluated with the help of the Euler–Maclaurin summation formula. This provides, *among the others*, a term given by the integral (the integer n being replaced by the continuous variable z)

$$\frac{J_{\mu\nu'}(s)}{\sqrt{\triangle}(x,x')} \sim \log(\sigma(x,x')/2)\int_0^{y^*} y\Gamma(-y-s)b_{y+1,\mu\nu'}dy, \tag{15}$$

with y^* in a small neighbourhood of the origin. On taking at last the $s \to 0$ limit we therefore recover the familiar $\log(\sigma(x,x'))$ singularity of the photon Green function, which results, ultimately, from non-vanishing Riemann curvature.

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4 Hadamard function: second covariant derivatives in the coincidence limit

Our Hadamard function $G_{\mu\nu'}^H$ is the imaginary part of the Feynman Green function in the formulae (11)–(14), and coincidence limits of its second covariant derivatives can be used to find the regularized energy-momentum tensor (see Christensen in Ref. [3]), since (the subscript $+$ denoting anti-commutators)

$$G_{\mu\nu'}^H = \langle [A_\mu, A_{\nu'}]_+ \rangle, \tag{16}$$

$$\begin{aligned}
\langle F_{\rho\gamma} F_{\tau\beta} \rangle = \lim_{x' \to x} \frac{1}{4} \Big(& G_{\gamma\beta';\rho\tau'}^H + G_{\beta\gamma';\tau\rho'}^H - G_{\gamma\tau';\rho\beta'}^H \\
& - G_{\tau\gamma';\beta\rho'}^H - G_{\rho\beta';\gamma\tau'}^H - G_{\beta\rho';\tau\gamma'}^H \\
& + G_{\rho\tau';\gamma\beta'}^H + G_{\tau\rho';\beta\gamma'}^H \Big),
\end{aligned} \tag{17}$$

$$\langle T^{\mu\nu} \rangle_{\text{Maxwell}} = \lim_{x' \to x} \left[\left(g^{\mu\rho} g^{\nu\tau} - \frac{1}{4} g^{\rho\tau} g^{\mu\nu} \right) g^{\gamma\beta} \langle F_{\rho\gamma} F_{\tau\beta} \rangle \right]. \tag{18}$$

In our investigation we find that the minimal-operator part of the Hadamard function contributes the divergent part (here square brackets $[...]$ denote the coincidence limits)

$$\left(\Gamma(0) \left\{ \left[b_1{}_{\gamma\beta';\rho\tau'} \right] - \frac{1}{6} \left[b_1{}_{\gamma\beta'} \right] R_{\rho\tau} \right\} - \frac{1}{2} \Gamma(-1) \left[b_2{}_{\gamma\beta'} \right] g_{\rho\tau} \right),$$

while the non-minimal operator part of the Hadamard function contributes further divergent terms given by

$$(\alpha - 1) \lim_{\varepsilon \to 0} \left(\frac{1}{\varepsilon} \Gamma(1) A_{\beta\gamma\rho\tau} + \Gamma(0) B_{\beta\gamma\rho\tau} + \frac{1}{2} \Gamma(-1) C_{\beta\gamma\rho\tau} \right),$$

where ε is here just a device to deal with $\frac{\delta_{n0}}{n}$ evaluated at $n = 0$, and

$$\begin{aligned}
A_{\beta\gamma\rho\tau} \equiv & \left[g_{\lambda\beta';}{}^\lambda{}_{\gamma\rho\tau'} \right] - \frac{1}{6} \left[g_{\lambda\beta';}{}^\lambda{}_\gamma \right] R_{\rho\tau} \\
& + \frac{1}{6} \left(\left[g_{\lambda\beta';}{}^\lambda{}_{\tau'} \right] R_{\gamma\rho} - \left[g_{\lambda\beta';}{}^\lambda{}_\rho \right] R_{\gamma\tau} + \left[g_{\lambda\beta';\gamma\tau'} \right] R^\lambda{}_\rho - \left[g_{\lambda\beta';\gamma\rho} \right] R^\lambda{}_\tau \right) \\
& + \left(\left[g_{\lambda\beta';\rho\tau'} \right] \frac{1}{6} R^\lambda{}_\gamma + \left[\sqrt{\triangle}_{;}{}^\lambda{}_{\gamma\rho\tau'} \right] g_{\lambda\beta} \right),
\end{aligned} \tag{19}$$

$$\begin{aligned}
B_{\beta\gamma\rho\tau} \equiv & \frac{1}{2} \left(\left[\bar{b}_1{}_{\lambda\beta';}{}^\lambda{}_{\tau'} \right] g_{\gamma\rho} - \left[\bar{b}_1{}_{\lambda\beta';}{}^\lambda{}_\rho \right] g_{\gamma\tau} \right) \\
& - \frac{1}{2} g_{\rho\tau} \left(\left[\bar{b}_1{}_{\lambda\beta';}{}^\lambda{}_\gamma \right] + \frac{1}{6} \left[\bar{b}_1{}_{\lambda\beta'} \right] R^\lambda{}_\gamma \right) \\
& - \frac{1}{12} \left(\left[\bar{b}_1{}_{\lambda\beta'} \right] (R^\lambda{}_\tau g_{\gamma\rho} + R^\lambda{}_\rho g_{\gamma\tau}) + \left[\bar{b}_1{}_{\tau\beta'} \right] R_{\gamma\rho} + \left[\bar{b}_1{}_{\rho\beta'} \right] R_{\gamma\tau} \right)
\end{aligned}$$

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$$+ \frac{1}{2} \left(\left[\overline{b}_{1 \ \rho\beta';\gamma\tau'} \right] - \left[\overline{b}_{1 \ \tau\beta';\gamma\rho} \right] \right)$$
$$- \frac{1}{6} \left(\left[\overline{b}_{1 \ \lambda\beta'} \right] (R^\lambda{}_{\rho\gamma\tau} + R^\lambda{}_{\tau\gamma\rho}) + \frac{1}{2} \left[\overline{b}_{1 \ \gamma\beta'} \right] R_{\rho\tau} \right), \tag{20}$$

$$C_{\beta\gamma\rho\tau} \equiv -2 \left(\left[\overline{b}_{2 \ \tau\beta'} \right] g_{\gamma\rho} + \left[\overline{b}_{2 \ \rho\beta'} \right] g_{\gamma\tau} \right) - \frac{1}{2} \left[\overline{b}_{2 \ \gamma\beta'} \right] g_{\rho\tau}. \tag{21}$$

We have therefore provided a covariant isolation of divergences resulting from every coincidence limit of second covariant derivatives of the Hadamard Green function. In our derivation of the results (19)–(21) we first sum over n, then take the $s \to 0$ limit and eventually evaluate the coincidence limit as $x' \to x$.

5   Summary

Our first original contribution is the expression (11)–(14) for the asymptotic expansion of the Feynman photon Green function at small values of the world function $\sigma(x, x')$. Our formulae provide a novel way of expressing the familiar $\frac{1}{\sigma}$ and $\log(\sigma)$ singularities in the local asymptotics; since they do not rely on the introduction of a mass term in the action functional, they are of particular interest for the reasons described in Sec. 1.

Moreover, we have obtained a formula for the coincidence limit of second covariant derivatives of the Hadamard function which can be used, in principle and also in practice, to obtain the Maxwell and gauge-fixing parts of the regularized energy-momentum tensor for arbitrary values of the gauge parameter α. In manifolds without boundary, the work of Endo [5] has shown that the trace anomaly resulting from the regularized $T^{\mu\nu}$ has the coefficient of the $\nabla^\mu \nabla_\mu R$ term which depends on the gauge parameter α. In manifolds with boundary, the integration of such a total divergence does not vanish, and further boundary invariants contribute to the regularized energy-momentum tensor. Thus, the calculation expressed by (19)–(21) is not of mere academic interest, but is going to prove especially useful when boundary effects are included. Of course, physical predictions are expected to be independent of α, but the actual proof is then going to be hard. More precisely, the work by Brown and Ottewill, [7] which differs from our approach because the $\alpha = 1$ case is there considered and the $\frac{1}{\sigma}$ and $\log(\sigma)$ singularities in the propagator are there assumed rather than derived, has been exploited by Allen and Ottewill [8] to show that, on using the Ward identity and the ghost wave equation, the energy-momentum tensor is α-independent up to geometric terms (i.e. up to polynomial expressions of dimension length^{-4} formed from the metric, the Riemann tensor and its covariant derivatives). The extension of these results to manifolds with boundary is, to our knowledge, an open research problem.

Acknowledgments

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